Beyond Euclidean Geometry

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A Wealth of Geometries

- So far, dealt with Euclidean geometry in 2 and 3 dimensions
- But a wealth of alternatives exist
  - Affine
  - Projective
  - Spherical
  - Inversive
  - Hyperbolic
  - Conformal
- Will look at all of these this afternoon!

What is a Geometry?

- A geometry consists of:
  - A set of objects (the elements)
  - A set of properties of these objects
  - A group of transformations which preserve these properties
- This is all fairly abstract!
- Used successfully in 19^{th} Century to unify a set of disparate ideas

Affine Geometry

- Points represented as displacements from a fixed origin
- Line through 2 points given by set

$$AB = a + \lambda(b - a)$$

- Affine transformation

$$t(x) = U(x) + a$$

- U is an invertible linear transformation
- As it stands, an affine transformation is not linear

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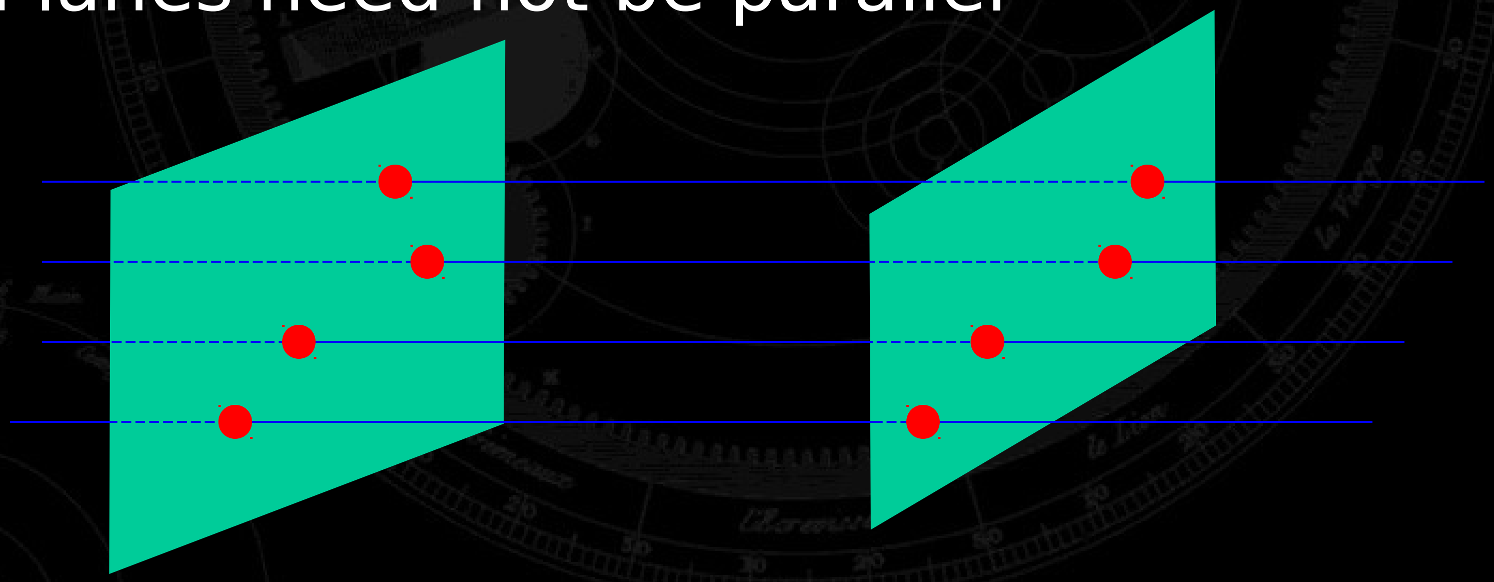
Parallel Lines

- Properties preserved under affine transformations:
  - Straight lines remain straight
  - Parallel lines remain parallel
  - Ratios of lengths along a straight line
- But lengths and angles are not preserved
- Any result proved in affine geometry is immediately true in Euclidean geometry

Geometric Picture

- Can view affine transformations in terms of parallel projections form one plane to another
- Planes need not be parallel

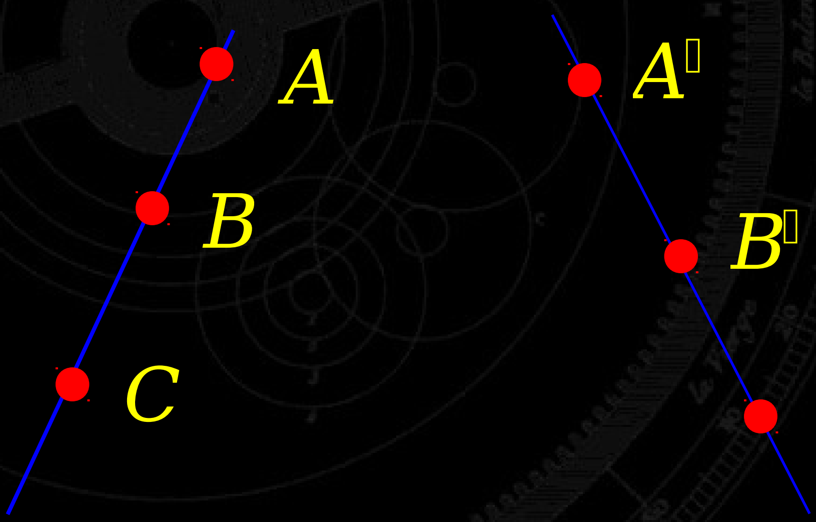


Line Ratios

- Ratio of distances along a line is preserved by an affine transformation

$$C = A + \alpha(B - A)$$

$$\frac{AC}{AB} = \frac{|\alpha(B - A)|}{|B - A|} = \alpha$$



$$C' = \mathrm{U}(A + \alpha(B - A)) + a$$
$$= A' + \alpha(B' - A')$$

Projective Geometry

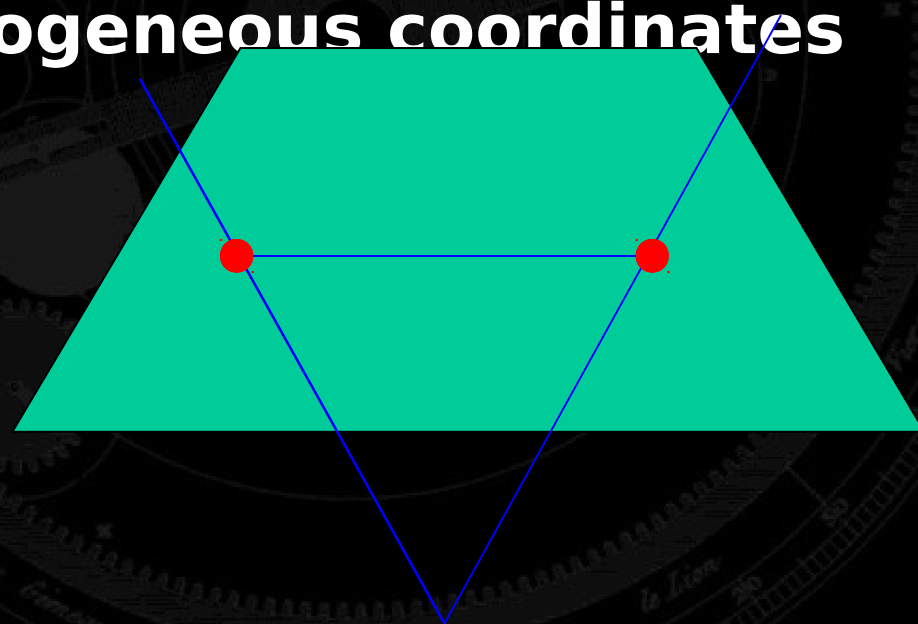
- Euclidean and affine models have a number of awkward features:
  - The origin is a special point
  - Parallel lines are special cases – they do not meet at a point
  - Transformations are not **linear**
- Projective geometry resolves all of these such that, for the plane
  - Any two points define a line
  - Any two lines define a point

The Projective Plane

- Represent points in the plane with lines in 3D
- Defines **homogeneous coordinates**

$$(x, y) \Rightarrow (a, b, c)$$

$$x = \frac{a}{c} \quad y = \frac{b}{c}$$



- Any multiple of ray represents same point

Projective Lines

- Points represented with grade-1 objects
- Lines represented with grade-2 objects
- If X lies on line joining A and B must have

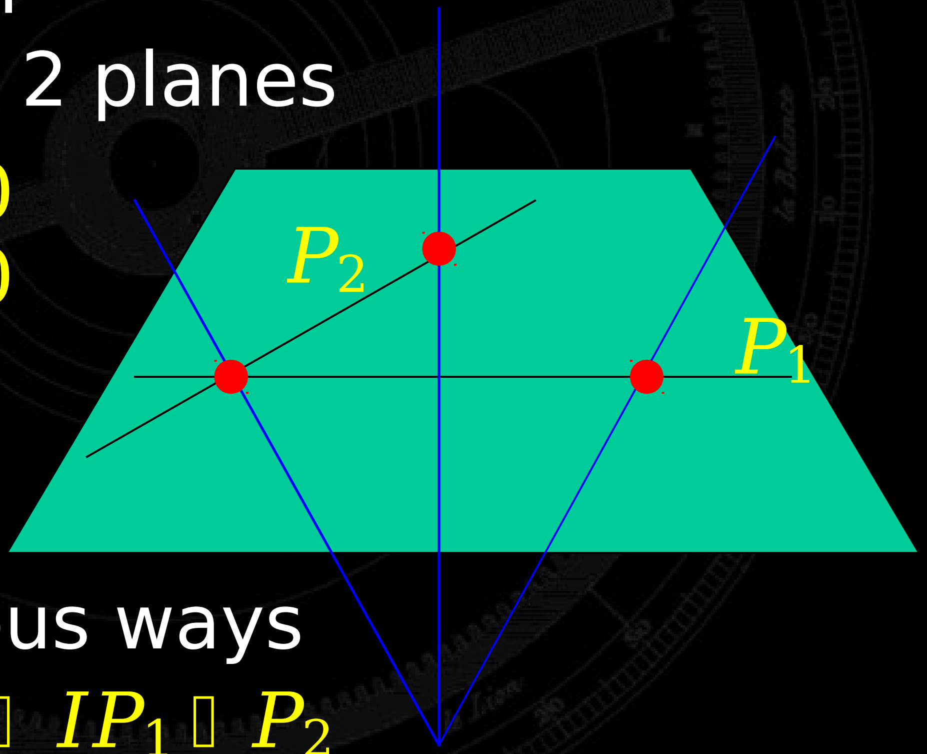
$$X \wedge A \wedge B = 0$$

- All info about the line encoded in the bivector $A \wedge B$
- Any two points define a line as a **blade**
- Can dualise this equation to $X \cdot n = 0 \quad n = IA \wedge B$

Intersecting Lines

- 2 lines meet at a point
- Need vector from 2 planes

$$X \square\ P_1 \square\ 0 \quad X \square\, p_1 \square\ 0$$
$$X \square\ P_2 \square\ 0 \quad X \square\, p_2 \square\ 0$$



- Solution

$$X \square\ I p_1 \square\ p_2$$

- Can write in various ways

$$X \square\ P_1 \square\, p_2 \square\ p_1 \square\, P_2 \square\ I P_1 \square\ P_2$$

Projective Transformations

- A general projective transformation takes

$$X \mapsto U(X)$$

- U is an invertible linear function
- Includes all affine transformations

$$\begin{pmatrix} x+a \\ y+b \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

- Linearises translations
- Specified by 4 points

Invariant Properties

- Collinearity and incidence are preserved by projective transformations

$$X \wedge A \wedge B \mapsto \mathsf{F}(X) \wedge \mathsf{F}(A) \wedge \mathsf{F}(B) = \mathsf{F}(X \wedge A \wedge B)$$

- This defines the notation on the right
- But these are all pseudoscalar quantities, so related by a multiple. In fact

$$\mathsf{F}(I) = \mathsf{F}(e_1) \wedge \mathsf{F}(e_2) \wedge \mathsf{F}(e_3) = \det(\mathsf{F})\, I$$

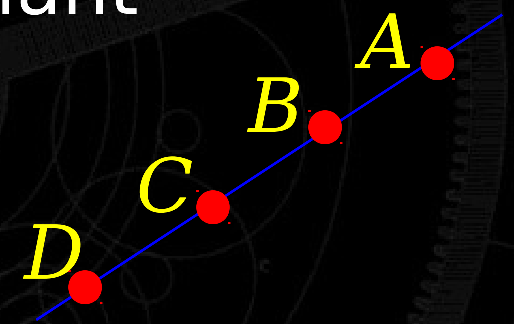
- So after the transformation

$$\mathsf{F}(X) \wedge \mathsf{F}(A) \wedge \mathsf{F}(B) = \det(\mathsf{F})\, X \wedge A \wedge B = 0$$

Cross Ratio

- Distances between 4 points on a line define a projective invariant

$$(ABCD) = \frac{AC\,DB}{AD\,CB}$$



- Recover distance using

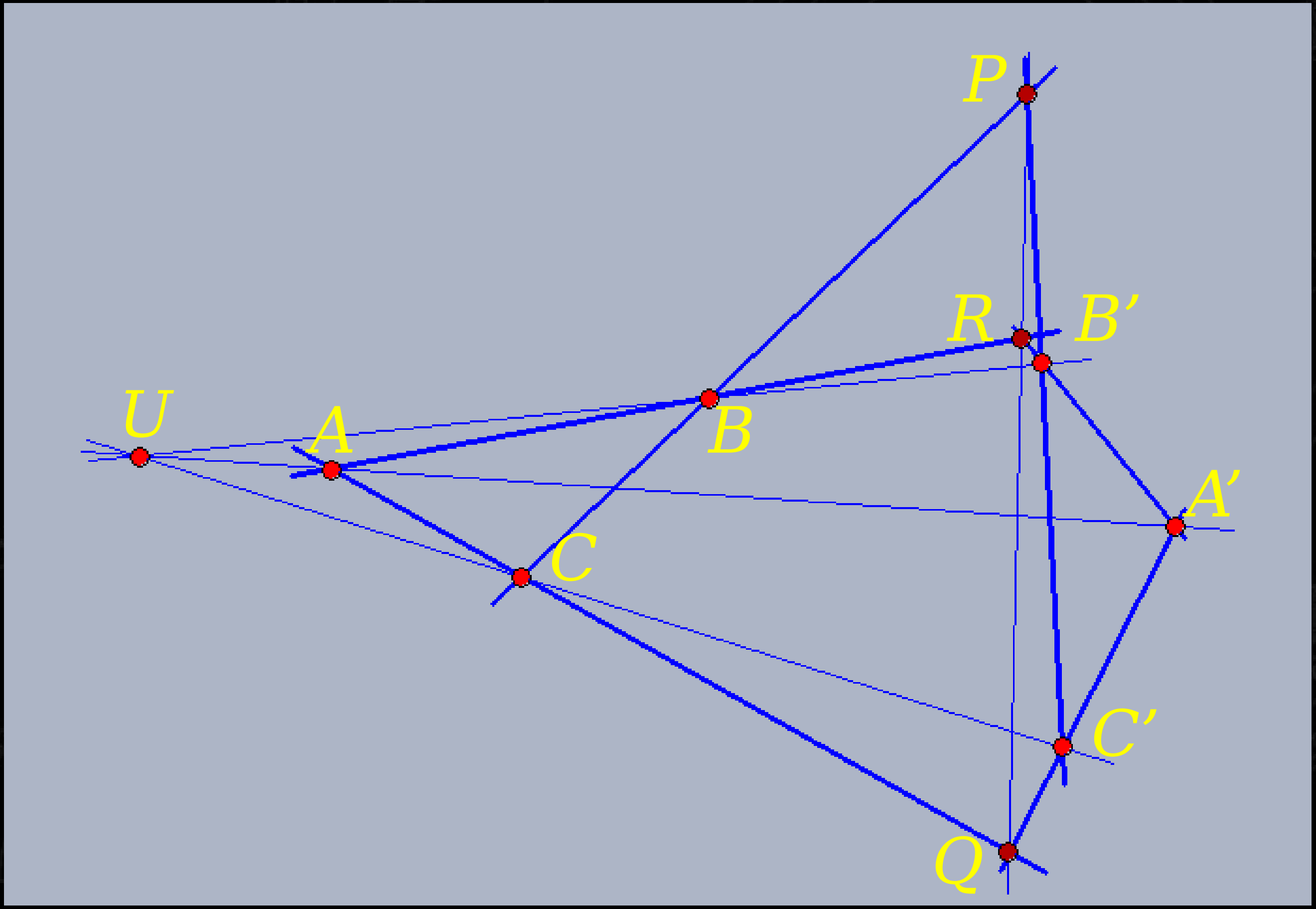
$$\frac{A}{A\cdot n} - \frac{B}{B\cdot n} = \frac{1}{A\cdot n\,B\cdot n}(A\wedge B)\cdot n$$

- Vector part cancels, so cross ratio is

$$\frac{A\wedge C\,D\wedge B}{A\wedge D\,C\wedge B}$$

Desargues' Theorem

- Two projectively related triangles



P, Q, R collinear

Figure produced using Cinderella

Proof

- Find scalars such that

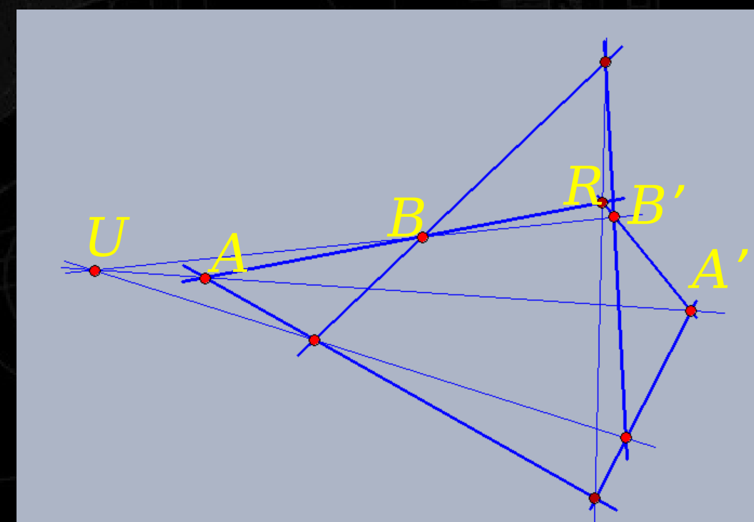
$$U \square \; \square A \square \; \square^{\square} A^{\square} \; \square \; \square B \square \; \square^{\square} B^{\square} \; \square \; \square C \square \; \square^{\square} C^{\square}$$

- Follows that

$$\square A \square \; \square B \; \square \; \square^{\square} B^{\square} \square \; \square^{\square} A^{\square} \; \square \; R$$

- Similarly

$$\square B \square \; \square C \; \square \; P \qquad \square C \square \; \square A \; \square \; Q$$



- Hence

$$P \square \; Q \square \; R \; \square \; 0 \qquad \square \qquad P \square \; Q \square \; R \; \square \; 0$$

3D Projective Geometry

- Points represented as vectors in 4D
- Form the 4D geometric algebra

$$1 \qquad e_i \qquad e_i e_j \qquad I e_i \qquad I$$

- 4 vectors, 6 bivectors, 4 trivectors and a pseudoscalar

$$I = e_1 e_2 e_3 e_4 \qquad I^2 = 1$$

- Use this algebra to handle points, lines and planes in 3D

Line Coordinates

- Line between 2 points A and B still given by bivector $A \wedge B$
- In terms of coordinates

$$(a + e_4) \wedge (b + e_4) = a \wedge b + (a - b) \wedge e_4$$

- The 6 components of the bivector define the **Plucker** coordinates of a line
- Only 5 components are independent due to constraint

$$(A \wedge B) \wedge (A \wedge B) = 0$$

Plane Coordinates

- Take outer product of 3 vectors to encode the plane they all lie in

$$P = A \wedge B \wedge C$$

- Can write equation for a plane as

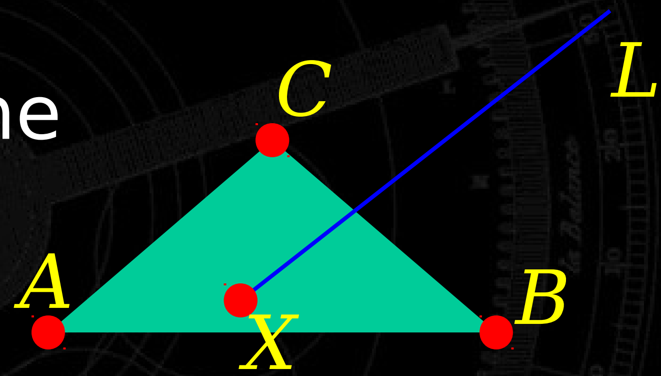
$$X \wedge P = 0 \qquad X \cdot (IP) = X \cdot p = 0$$

- Points and planes related by duality
- Lines are dual to other lines
- Use geometric product to simplify expressions with inner and outer products

Intersections

- Typical application is to find intersection of a line and a plane

$$X = (A \wedge B \wedge C) \vee L$$



- Replace meet with duality

$$X = (I A \wedge B \wedge C) \cdot (I L) I = \boxed{p \cdot L}$$

- Where $p = I A \wedge B \wedge C$
- Note the non-metric use of the inner product

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Intersections II

- Often want to know if a line cuts within a chosen simplex
- Find intersection point and solve

$$X = p \wedge L = \alpha A + \beta B + \gamma C$$

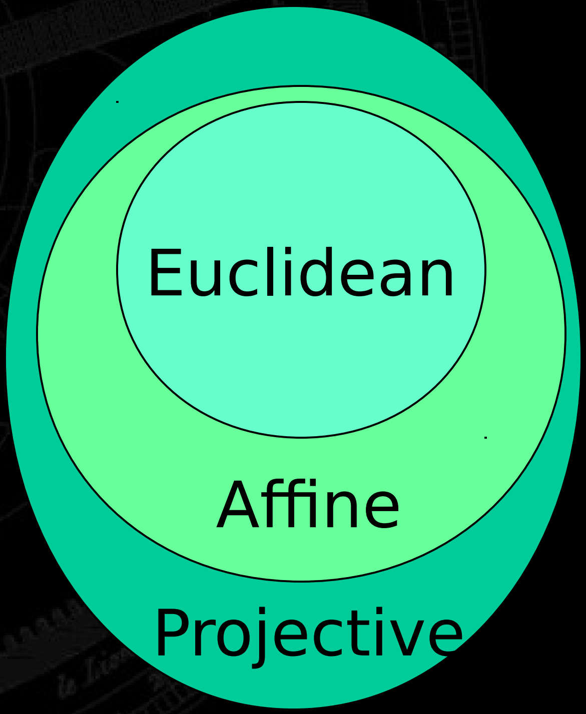
- Rescale all vectors so that 4th component is 1

$$\alpha + \beta + \gamma = 1$$

- If all of α, β, γ are positive, the line intersects the surface within the simplex

Euclidean Geometry Recovered

- Affine geometry is a subset of projective geometry
- Euclidean geometry is a subset of affine geometry
- How do we recover Euclidean geometry from projective?
- Need to find a way to impose a distance measure



Fundamental Conic

- Only distance measure in projective geometry is the cross ratio
- Start with 2 points and form line through them
- Intersect this line with the **fundamental conic** to get 2 further points X and Y
- Form cross ratio $r = \frac{A - X \, B - Y}{A - Y \, B - X}$
- Define distance by $d = \ln(r)$

Cayley-Klein Geometry

- Cayley & Klein found that different fundamental conics would give Euclidean, spherical and hyperbolic geometries
- United the main classical geometries
- But there is a major price to pay for this unification:
  - All points have complex coordinates!
- Would like to do better, and using GA we can!

Further Information

- All papers on Cambridge GA group website: www.mrao.cam.ac.uk/~clifford
- Applications of GA to computer science and engineering are discussed in the proceedings of the AGACSE 2001 conference. www.mrao.cam.ac.uk/agacse2001
- IMA Conference in Cambridge, 9th Sept 2002
- ‘Geometric Algebra for Physicists’ (Doran + Lasenby). Published by CUP, soon.